

# A COMBINATORIAL PROPERTY OF IDEALS IN FREE PROFINITE MONOIDS

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The reader is referred to [8] for all undefined notation concerning finite and profinite semigroups. We assume throughout this note that  $\mathbf{V}$  is a pseudovariety of monoids [1, 5, 8] closed under Mal'cev product with the pseudovariety  $\mathbf{A}$  of aperiodic monoids, i.e.,  $\mathbf{A} \boxtimes \mathbf{V} = \mathbf{V}$ . Denote by  $\hat{F}_{\mathbf{V}}(A)$  the free pro- $\mathbf{V}$  monoid on a profinite space  $A$  [1, 3, 8]. In this note we prove the following theorem.

**Theorem 1.** *Suppose that  $\alpha_1, \dots, \alpha_m \in \hat{F}_{\mathbf{V}}(A)$  and  $I_1, \dots, I_n$  are closed ideals in  $\hat{F}_{\mathbf{V}}(A)$  where  $m \leq n$ . If  $\alpha_1 \cdots \alpha_m \in I_1 \cdots I_n$ , then  $\alpha_i \in I_j$  for some  $i$  and  $j$ .*

Before proving the theorem, we state a number of consequences. Recall that an ideal  $I$  in a semigroup is *prime* if  $ab \in I$  implies  $a \in I$  or  $b \in I$ .

**Corollary 2.** *Let  $I = I^2$  be a closed idempotent ideal of  $\hat{F}_{\mathbf{V}}(A)$ . Then  $I$  is prime.*

*Proof.* Suppose that  $ab \in I = I^2$ . Then  $a \in I$  or  $b \in I$  by Theorem 1.  $\square$

An element  $a$  of a semigroup  $S$  is said to be regular if there exists  $b \in S$  so that  $aba = a$ . Any regular element of a profinite semigroup generates a closed idempotent ideal. Hence we have:

**Corollary 3.** *Every regular element of  $\hat{F}_{\mathbf{V}}(A)$  generates a prime ideal. In particular, the minimal ideal of  $\hat{F}_{\mathbf{V}}(A)$  is prime.*

The second statement of Corollary 3 was first proved by Almeida and Volkov using techniques coming from symbolic dynamics [2].

Our next result generalizes a result of Rhodes and the author showing that all elements of finite order in  $\hat{F}_{\mathbf{V}}(A)$  are group elements, which played a key role in proving that such elements are in fact idempotent [7]. Let  $\hat{\mathbb{N}}$  denote the profinite completion of the monoid of natural numbers; it is in fact a profinite semiring. We use  $\omega$  for the non-zero idempotent of  $\hat{\mathbb{N}}$ .

**Corollary 4.** *Let  $\alpha \in \hat{F}_{\mathbf{V}}(A)$  satisfy  $a^n = a^{n+\lambda}$  for some positive integer  $n$  and some  $0 \neq \lambda \in \hat{\mathbb{N}}$ . Then  $a$  is a group element, i.e.,  $a = a^\omega a$ .*

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*Proof.* It is immediate that  $a^n = a^{n+k\lambda}$  for all  $k \in \mathbb{N}$  and hence all  $k \in \widehat{\mathbb{N}}$ . Thus  $a^n = a^{n+\omega\lambda} = a^\omega$ . Since  $a^\omega$  is regular, it generates a prime ideal by Corollary 2. It follows that  $a \not\in a^\omega$  and hence  $a \in \mathcal{H} a^\omega$ . Thus  $a$  is a group element, as required.  $\square$

To prove the theorem, we use the Henckell-Schützenberger expansion. Let  $M$  be a finite monoid generated by a set  $A$ . For any element  $\alpha \in \widehat{F}_{\mathbf{V}}(A)$ , we write  $[\alpha]_M$  for its image in  $M$ . For  $w \in A^*$  (the free monoid on  $A$ ), define  $\text{cut}_n(w)$  to be the set of all  $n$ -tuples  $(m_1, \dots, m_n)$  of  $M$  such that there exists a factorization  $w = w_1 \cdots w_n$  with  $[w_i]_M = m_i$ , for  $i = 1, \dots, n$ . It is well known that the equivalence relation on  $A^*$  given by  $u \sim v$  if  $\text{cut}_n(u) = \text{cut}_n(v)$  is a congruence of finite index contained in the kernel of the natural map  $A^* \rightarrow M$  [4, 6]. Moreover, if we denote by  $M^{(n)}$  the quotient  $A^*/\sim$ , then the natural map  $\eta: M^{(n)} \rightarrow M$  is aperiodic and hence if  $M \in \mathbf{V}$ , then  $M^{(n)} \in \mathbf{V}$  under our hypothesis on  $\mathbf{V}$  [4, 6].

Our main theorem relies on the following simple factorization lemma for free monoids.

**Lemma 5.** *Let  $w \in A^*$  and suppose  $w = u_1 \cdots u_m = v_1 \cdots v_n$  where  $m \leq n$ . Then  $v_j$  is a factor of  $u_i$  for some  $1 \leq i \leq m$  and  $1 \leq j \leq n$ .*

*Proof.* If any  $v_j$  is empty, we are done so assume now each  $v_j$  is non-empty and hence  $w$  is non-empty. If any  $u_i$  is empty we may omit it, so assume  $u_i$  is non-empty for  $i = 1, \dots, m$ . Define a function  $f: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$  by  $f(j) = i$  if the last letter of  $v_j$  belongs to  $u_i$ . The map  $f$  is monotone. Suppose first that  $f$  is not injective. Then there exists  $2 \leq j \leq n$  so that  $f(j-1) = f(j)$ . In this case,  $v_j$  is a factor of  $u_i$  where  $i = f(j)$ . If  $f$  is injective, then since it is monotone and  $m \leq n$ , we must have that  $m = n$  and  $f$  is the identity map. In this case,  $u_1$  is a factor of  $v_1$ .  $\square$

*Proof of Theorem 1.* Suppose first that  $A$  is finite and that  $\alpha_2, \dots, \alpha_m \notin I_j$  for any  $1 \leq j \leq n$  and that  $\alpha_1 \notin I_2, \dots, I_n$ . We show that  $\alpha_1 \in I_1$ . Since  $I_1$  is closed, it suffices to show that  $\pi(\alpha_1) \in \pi(I_1)$  for all continuous surjective homomorphisms  $\pi: \widehat{F}_{\mathbf{V}}(A) \rightarrow V$  with  $V \in \mathbf{V}$ . Using that  $I_1, \dots, I_n$  are closed we may assume without loss of generality  $\pi(\alpha_i) \notin I_j$  for  $2 \leq i \leq m$  and  $1 \leq j \leq n$ , or  $i = 1$  and  $2 \leq j \leq n$ . By assumption  $\alpha_1 \cdots \alpha_m \in I_1 \cdots I_n$  so there exist  $\beta_j \in I_j$ , for  $j = 1, \dots, n$ , so that  $\alpha_1 \cdots \alpha_m = \beta_1 \cdots \beta_n$ .

Since  $A^*$  is dense in  $\widehat{F}_{\mathbf{V}}(A)$ , we can find words  $u_1, \dots, u_m \in A^*$  so that  $[u_i]_{V^{(n)}} = [\alpha_i]_{V^{(n)}}$ , for  $i = 1, \dots, m$  and words  $w_1, \dots, w_n$  so that  $[w_j]_{V^{(n)}} = [\beta_j]_{V^{(n)}}$  for  $j = 1, \dots, n$ . Then

$$[u_1 \cdots u_m]_{V^{(n)}} = [\alpha_1 \cdots \alpha_m]_{V^{(n)}} = [\beta_1 \cdots \beta_n]_{V^{(n)}} = [w_1 \cdots w_n]_{V^{(n)}}$$

and so there exists a factorization  $u_1 \cdots u_m = v_1 \cdots v_n$  so that  $[v_j]_V = [w_j]_V$ , for  $i = 1, \dots, n$ . By Lemma 5 it follows that there exist  $i$  and  $j$  so that  $v_j$  is a factor of  $u_i$ . Since  $[v_j]_V = [w_j]_V = [\beta_j]_V \in \pi(I_j)$ , it follows that  $\pi(\alpha_i) = [u_i]_V \in \pi(I_j)$ . By our assumption on  $\pi$ , it follows that  $i = 1 = j$ . Thus  $\pi(\alpha_1) \in \pi(I_1)$ , as required. This completes the proof when  $A$  is finite.

Suppose next that  $A$  is profinite; so  $A = \varprojlim_{d \in D} A_d$  with  $D$  a directed set and  $A_d$  finite for  $d \in D$ . Then (cf. [1, 3]) one has  $\widehat{F}_{\mathbf{V}}(A) = \varprojlim_{d \in D} \widehat{F}_{\mathbf{V}}(A_d)$ . Suppose now that  $\alpha_2, \dots, \alpha_m \notin I_j$  for any  $1 \leq j \leq n$  and  $\alpha_1 \notin I_2, \dots, I_n$ . We show that  $\alpha_1 \in I_1$ . Since  $I_1$  is closed, it suffices to prove, for all  $d \in D$ , that  $\pi_d(\alpha_1) \in \pi_d(I_1)$  where  $\pi_d: \widehat{F}_{\mathbf{V}}(A) \rightarrow \widehat{F}_{\mathbf{V}}(A_d)$  is the canonical projection.

Now  $\pi_d(I_j)$  is a closed ideal for  $1 \leq j \leq n$ , so we may assume without loss of generality that  $\pi_d(\alpha_i) \notin I_j$  for  $2 \leq i \leq m$  and  $1 \leq j \leq n$ , or  $i = 1$  and  $2 \leq j \leq n$ . By the previous case,  $\pi_d(\alpha_i) \in \pi_d(I_j)$  for some  $i$  and  $j$ . By assumption we must have  $i = 1$  and  $j = 1$ . This completes the proof.  $\square$

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